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# The tensor radiative transfer equation

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**Abstract.** The vector radiative transfer equation is used for the solution of numerous problems in the field of random media optics. It describes the transformation of the Stokes vector of a light beam due to its propagation and scattering in a medium. However, the Stokes vector depends on the choice of a coordinate system. Thus, one needs to account for the rotation of coordinate systems within the framework of the standard formulation of the vector radiative transfer equation. The coordinate-free approach has advantages in the case of complex random media, including anisotropic and chiral ones. The aim of this paper is to derive the coordinate-invariant tensor radiative transfer equation. The derivation is based on the concept of the light beam tensor. We also study the relationship between the tensor and vector forms of the radiative transfer equation.

## 1. Introduction

The propagation of light in turbid media is often described within the framework of vector radiative transfer theory [1–3]. The vector radiative transfer equation [1–3] describes the change of the Stokes vector $\vec{S}$ of a light beam due to its propagation and scattering in a random medium. The components of the Stokes vector $I$, $Q$, $U$ and $V$ can be related to the components of the electric vector $\vec{E} = E_1\vec{l}_1 + E_2\vec{l}_2$ of a plane monochromatic wave by the following equations:

$$\begin{aligned} I &= |E_1|^2 + |E_2|^2 \qquad & Q &= |E_1|^2 - |E_2|^2 \\ U &= 2\,\mathrm{Re}\left(E_1 E_2^*\right) \qquad & V &= -2\,\mathrm{Im}\left(E_1 E_2^*\right) \end{aligned} \tag{1}$$

where the values of $E_1$ and $E_2$ are components of the electric field in the plane, perpendicular to the direction of propagation. We have omitted the common multiplier in equations (1) for the sake of simplicity. It should be pointed out that equations (1) only hold for monochromatic waves. In the more general case of non-monochromatic waves, the phases and amplitudes of the electric vector are not constant and the parameters (1) should be averaged with respect to time $t$.

One can see that the Stokes vector is defined within the framework of the special coordinate system, which is attached to the direction of light propagation. This direction is changed many times during the process of light scattering in a random medium. This feature is accounted for by using special rotation matrices [2, 3]. It is of general importance to derive the radiative transfer equation in a tensor form, which does not depend on the specific coordinate system.

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The covariant methods were introduced into the optics of anisotropic and gyrotropic media by Fedorov [4, 5] and led to rapid progress in the field of anisotropic media optics [4–6]. This was due to the fact that the direct manipulation of vectors, dyadics and their invariants facilitates solutions, condenses exposition and provides results of greater generality.

However, the coordinate-free approach has not been used much in the optics of random anisotropic media so far. The aim of this paper is to derive the covariant tensor radiative transfer equation and bring attention to covariant methods in radiative transfer theory.

## 2. Light beam tensor

Let us define the light beam as a superposition of different incoherent simple uniform waves. These waves can have random phases and different states of polarizations. However, they are characterized by the same speed and direction of propagation. The general case of incoherent light beams may be studied using the notion of the light beam tensor [5]

$$F = \sum_s \vec{E}^{(s)}\vec{E}^{(s)*} \qquad \vec{n}\vec{E}^{(s)} = \vec{n}\vec{E}^{(s)*} = 0 \tag{2}$$

where $\vec{E} = E_1\vec{x}+E_2\vec{y}+E_3\vec{z}$ is the electric vector, $\sum_s$ denotes the summation over all incoherent simple waves in a beam, and the vector $\vec{n}$ defines the direction of propagation. The dyadic notation $F = \vec{E}\vec{E}^*$ means

$$F = \begin{pmatrix} E_1E_1^* & E_1E_2^* & E_1E_3^* \\ E_2E_1^* & E_2E_2^* & E_2E_3^* \\ E_3E_1^* & E_3E_2^* & E_3E_3^* \end{pmatrix}. \tag{3}$$

One obtains within the framework of the coordinate system, attached to the direction of propagation $\vec{n}\|\vec{z}$ that $E_3 = 0$ and the tensor $F$ (3) reduces to the density matrix [3, 5]

$$\rho = \begin{pmatrix} |E_1|^2 & E_1E_2^* \\ E_2E_1^* & |E_2|^2 \end{pmatrix}. \tag{4}$$

The elements of the matrix $\rho$ can be expressed in terms of components of the Stokes vector $\vec{S}$ (see equation (1))

$$\rho = \frac{1}{2}\begin{pmatrix} I+Q & U-\mathrm{i}V \\ U+\mathrm{i}V & I-Q \end{pmatrix}. \tag{5}$$

Thus, both (1) and (4) depend on the choice of the coordinate system. One can find the radiative transfer equation for the density matrix $\rho$ in [3].

Fedorov [5] introduced the following invariants of the light beam tensor $F$:

$$I = F_t \qquad K = \left(F^2\right)_t \qquad M = \mathrm{i}\left(n^\times F\right)_t \qquad L = \left(FF^*\right)_t \tag{6}$$

where $t$ denotes the trace (e.g. $F_t = |E_1|^2+|E_2|^2+|E_3|^2$, see equation (3)) and $n^\times$ is the tensor with components

$$n^\times_{ac} = e_{abc}n_b. \tag{7}$$

Here values of $n_b$ are components of the vector $\vec{n}$ and values $e_{abc}$ compose the Levi-Civita tensor [5] with components $e_{123} = e_{231} = e_{312} = 1$ and $e_{213} = e_{132} = e_{321} = -1$. Other components of the Levi-Civita tensor are equal to zero. Equation (7) can be written in the following explicit form:

$$n^{\times} = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}.$$

It follows from equation (6) that

$$\begin{aligned} I &= J_1 + J_2 + J_3 \\ K &= J_1^2 + J_2^2 + J_3^2 + 2\,(F_{12}F_{21} + F_{13}F_{31} + F_{23}F_{32}) \\ L &= J_1^2 + J_2^2 + J_3^2 + F_{12}^2 + F_{21}^2 + F_{13}^2 + F_{31}^2 + F_{23}^2 + F_{32}^2 \\ M &= \mathrm{i}\,[n_3\,(F_{12} - F_{21}) + n_2\,(F_{31} - F_{13}) + n_1\,(F_{23} - F_{32})] \end{aligned}$$

where $F_{ij} = \sum_s E_i^s E_j^{s*}$, $J_j = F_{jj}$ and $\sum_s$ denotes the sum over rays in a beam.

It follows that

$$\begin{aligned} I &= J_1 + J_2 & K &= J_1^2 + J_2^2 + 2F_{12}F_{21} \\ L &= J_1^2 + J_2^2 + F_{12}^2 + F_{21}^2 & M &= 2\,\mathrm{Im}(F_{12}) \end{aligned}$$

at $\vec{n}\|\vec{z}(n_1 = n_2 = 0, n_3 = 1)$. One can see that the first invariant describes the intensity of the light beam $I$. The invariant $M$ at $\vec{n}\|\vec{z}$ coincides with the fourth component of the Stokes vector $V$ (see equation (1)). Thus, the first and last components of the Stokes vectors (see equation (1)) are coordinate independent. It should be pointed out that this is not a case for components $Q$ and $U$.

The ratio $K/I^2$ determines the degree of polarization [5]

$$p = \sqrt{\frac{2K}{I^2} - 1} \tag{8}$$

and the invariant $L$ can be used to find the semi-axes of the polarization ellipse [5]

$$a = \sqrt{\tfrac{1}{2}(I + \sqrt{L})} \qquad b = \sqrt{\tfrac{1}{2}(I - \sqrt{L})}. \tag{9}$$

For instance, it follows that

$$a = b = \sqrt{I/2} \tag{10}$$

at $L = 0$. This is the case for circularly polarized light [5].

Values (1) can be found from components of the tensor $F$. Thus, one can see that the light beam tensor and its invariants can be used for the complete description of the polarization characteristics of a light beam.

## 3. The radiative transfer equation

Let us now find the transfer equation for the light beam tensor in a random medium. It follows on general grounds that the change of the light beam tensor $\mathrm{d}F\,(\vec{n})$ in the direction, specified by the vector $\vec{n}$, is due to two processes, namely, due to light scattering $(\mathrm{d}F^{(1)}\,(\vec{n}))$ on the path $\mathrm{d}n$ from all directions to the direction $\vec{n}$, and light extinction $(\mathrm{d}F^{(2)}\,(\vec{n}))$ on the path $\mathrm{d}n$ due to beam propagation in a light scattering and absorbing medium

$$\mathrm{d}F = \mathrm{d}F^{(1)} + \mathrm{d}F^{(2)}. \tag{11}$$

Let us now calculate these contributions for a cylindrical volume with a unit cross section. First of all we note that light scattered by a single particle can be presented as a spherical wave

$$\vec{E}_{\rm sca}(\vec{p})\frac{\exp(\mathrm{i}kR)}{R}$$

where $\vec{E}_{\rm sca}(\vec{p})$ is the vector amplitude of the spherical wave scattered in the direction $\vec{p}$, $R$ is the distance to the observation point and $k$ is the wavenumber. The amplitude $\vec{E}_{\rm sca}(\vec{p})$ can be presented in the following form due to the linearity of the Maxwell equations:

$$\vec{E}_{\rm sca}(\vec{p}) = f(\vec{p},\vec{q})\vec{E}_0(\vec{q}) \tag{12}$$

where $\vec{E}_0(\vec{q})$ is the electric field of the incident wave, propagating in the direction specified by the vector $\vec{q}$, and $f(\vec{p},\vec{q})$ is the scattering tensor. Let us introduce the light beam tensor of the incident beam

$$F = \sum_s E_0^{(s)}(\vec{q})E_0^{(s)^*}(\vec{q}) \tag{13}$$

and the light beam tensor of the scattered light

$$F' = \sum_s E_{\rm sca}^{(s)}(\vec{p})E_{\rm sca}^{(s)^*}(\vec{p}) \tag{14}$$

where the symbol $s$ represents a single ray in a beam. It follows from equations (12)–(14) that

$$F'(\vec{p}) = \sum_s \vec{E}_{\rm sca}^{(s)}\vec{E}_{\rm sca}^{*(s)} = \sum_s f\vec{E}_0^{(s)}\vec{E}_0^{*(s)}f^+ = fF(\vec{q})f^+ \tag{15}$$

where $f^+ = \tilde{f}^*$ and $\tilde{f}$ is the transpose ($\tilde{f}_{ik} = f_{ki}$) tensor. This formula establishes the law of transformation of the light beam tensor due to the scattering process from the direction $\vec{q}$ to the direction $\vec{p}$. It is clear that the total light scattering in the direction specified by the vector $\vec{n}$ on the path d$n$ from all directions $\vec{n}'$ will differ from equation (15) due to the integration on the solid angle $\mathrm{d}\Omega_{\vec{n}'}$, namely

$$\mathrm{d}F^{(1)}(\vec{n}) = N\,\mathrm{d}n\int \mathrm{d}\Omega_{\vec{n}'}\, f(\vec{n}',\vec{n})F(\vec{n}')f^+(\vec{n}',\vec{n})$$

where $N$ is the number of scattering events in a unit volume.

The removal of photons from the direction $\vec{n}$ on the path d$n$ due to the extinction process can be described by the operator $\hat{L}$

$$\mathrm{d}F^{(2)}(\vec{n}) = -N\hat{L}F(\vec{n})\,\mathrm{d}n. \tag{16}$$

Thus, it follows from equation (11) that

$$\frac{\mathrm{d}F(\vec{n})}{\mathrm{d}n} = -N\hat{L}F(\vec{n}) + N\int \mathrm{d}\Omega_{\vec{n}'}\, f(\vec{n}',\vec{n})F(\vec{n}')f^+(\vec{n}',\vec{n}) + B(\vec{n}) \tag{17}$$

where the term $B(\vec{n})$ describes the internal emitting sources. This is the radiative transfer equation written within the framework of the coordinate-free approach.

Let us now determine the operator $\hat{L}$. The tensor $F$ can be presented in the following form:

$$F(\vec{n}) = F_{\rm c}(\vec{n})\delta(\vec{n}-\vec{n}_0) + F_{\rm d}(\vec{n}) \tag{18}$$

where $\delta(\vec{n}-\vec{n}_0)$ is the delta function. Values $F_{\rm c}(\vec{n})$ and $F_{\rm d}(\vec{n})$ describe the direct (coherent) and diffused (incoherent) parts of the light field in a random medium, respectively. The direction of the vector $\vec{n}_0$ coincides with the direction of an incident beam. It follows from equations (17) and (18) that

$$\dot{F}_{\rm c}(\vec{n}_0) + \dot{F}_{\rm d}(\vec{n}) = -N\hat{L}F_{\rm c}(\vec{n}_0) - N\hat{L}F_{\rm d}(\vec{n}) + N\int {\rm d}\Omega_{\vec{n}'}\, f(\vec{n}',\vec{n})F_{\rm d}(\vec{n}')f^{+}(\vec{n}',\vec{n})$$
$$+Nf(\vec{n}_0,\vec{n})F_{\rm c}(\vec{n}_0)f^{+}(\vec{n}_0,\vec{n}) + B(\vec{n}) \qquad (19)$$

where

$$\dot{F}(\vec{n}) \equiv \frac{{\rm d}F}{{\rm d}n} = (\vec{n}\cdot\vec{\nabla})F.$$

Thus, one can obtain

$$\dot{F}_{\rm c}(\vec{n}_0) = -N\hat{L}F_{\rm c}(\vec{n}_0) \qquad (20)$$

for the coherent component and

$$\dot{F}_{\rm d}(\vec{n}) = -N\hat{L}F_{\rm d}(\vec{n}) + N\int {\rm d}\Omega_{\vec{n}'}\, f(\vec{n}',\vec{n})F_{\rm d}(\vec{n}')f^{+}(\vec{n}',\vec{n}) + B_0(\vec{n}) + B(\vec{n}) \qquad (21)$$

for the diffused light field. The tensor

$$B_0(\vec{n}) = Nf(\vec{n}_0,\vec{n})F_{\rm c}(\vec{n}_0)f^{+}(\vec{n}_0,\vec{n}) \qquad (22)$$

describes the single light scattering of an incident light beam. It can be easily obtained after calculation of $F_{\rm c}(\vec{n}_0)$ from equation (20).

It is well known that the coherent field $\vec{E}_{\rm c}$ propagating in the direction $\vec{n}_0$ satisfies the following formula [7]:

$$\dot{\vec{E}}_{\rm c}(\vec{n}_0) = {\rm i}\lambda Nf(\vec{n}_0,\vec{n}_0)\vec{E}_{\rm c}(\vec{n}_0) \qquad (23)$$

where $\lambda$ is the wavelength.

From equation (23) and the definition of the coherent component of the light beam tensor

$$F_{\rm c} = \sum_{s} \vec{E}_{\rm c}^{s}\vec{E}_{\rm c}^{s^*} \qquad (24)$$

one obtains

$$\dot{F}_{\rm c} = \sum_{s} \dot{\vec{E}}_{\rm c}^{s}\vec{E}_{\rm c}^{s^*} + \vec{E}_{\rm c}^{s}\dot{\vec{E}}_{\rm c}^{s^*} = {\rm i}\lambda N\sum_{s}\left(f\vec{E}_{\rm c}^{s}\vec{E}_{\rm c}^{s^*} - \vec{E}_{\rm c}^{s}f^{*}\vec{E}_{\rm c}^{s^*}\right) \qquad (25)$$

or

$$\dot{F}_{\rm c} = {\rm i}\lambda N(fF_{\rm c} - F_{\rm c}f^{+}). \qquad (26)$$

It follows from equations (20) and (26) for the operator $\hat{L}$ that

$$\hat{L}\,|\psi\rangle = -{\rm i}\lambda\left(f\,|\psi\rangle - \langle\psi|\,f^{+}\right) \qquad (27)$$

where we have used Dirac notation.

Finally, we obtain the covariant radiative transfer equation (CRTE) for the diffused light (see equations (21) and (22))

$$(\vec{n}\cdot\vec{\nabla})F_{\rm d}(\vec{r},\vec{n}) = {\rm i}\lambda N\left[f(\vec{n},\vec{n})F_{\rm d}(\vec{r},\vec{n}) - F_{\rm d}(\vec{r},\vec{n})f^{+}(\vec{n},\vec{n})\right]$$
$$+N\int {\rm d}\Omega_{\vec{n}'}\, f(\vec{n}',\vec{n})F_{\rm d}(\vec{r},\vec{n}')f^{+}(\vec{n}',\vec{n})$$
$$+Nf(\vec{n}_0,\vec{n})F_{\rm c}(\vec{r},\vec{n}_0)f^{+}(\vec{n}_0,\vec{n}) + B(\vec{n}) \qquad (28)$$

where $\vec{r}$ is the radius vector of the observation point.

The value of $F_c(\vec{r}, \vec{n}_0)$ in equation (28) is determined from (26), which can be written in the following form:

$$(\vec{n} \cdot \vec{\nabla}) F_c = i\lambda N\big(f(\vec{n}_0, \vec{n}_0) F_c - F_c f^+(\vec{n}_0, \vec{n}_0)\big) \tag{29}$$

or

$$F_c(\vec{r}, \vec{n}_0) = \exp\{-Nh\hat{L}\} F_0(\vec{r}, \vec{n}_0) \tag{30}$$

where $F_0(\vec{n}_0)$ is the beam tensor for the incident light and $h$ is the length of a beam path in a medium, which we consider to be uniform. Equations (27) and (30) provide the generalization of the Bouger law.

The boundary conditions for equation (28) state that there is no diffused light arriving at a scattering convex medium from outside,

$$F_d(\vec{r}_0, \vec{n}) = 0 \qquad (\text{at } \vec{n} \cdot \vec{l} < 0) \tag{31}$$

where $\vec{l}$ is the unit vector normal to the boundary in the outward direction at the point with the radius vector $\vec{r}_0$.

Note that the general (without introducing separate coherent and diffused components) CRTE can be obtained from equations (17) and (27):

$$(\vec{n} \cdot \vec{\nabla}) F(\vec{r}, \vec{n}) = i\lambda N[f(\vec{n}, \vec{n}) F(\vec{r}, \vec{n}) - F(\vec{r}, \vec{n}) f^+(\vec{n}, \vec{n})] + N \int d\Omega_{\vec{n}'}\, f(\vec{n}', \vec{n}) F(\vec{r}, \vec{n}) f^+(\vec{n}', \vec{n}) + B(\vec{n}) \tag{32}$$

with the boundary condition

$$F(\vec{r}_0, \vec{n}) = F_0(\vec{r}_0, \vec{n}) \qquad (\text{at } \vec{n} \cdot \vec{l} < 0)$$

where $\vec{r}_0$ is the radius vector of the point at the boundary of a scattering medium.

Equation (32) can be transformed into the standard vector radiative transfer equation, assuming that $\vec{n} \| \vec{z}$. It follows from equation (32) that in this case

$$(\vec{n} \cdot \vec{\nabla}) \rho(\vec{r}, \vec{n}) = i\lambda N \left[A(\vec{n}, \vec{n}) \rho(\vec{r}, \vec{n}) - \rho(\vec{r}, \vec{n}) A^+(\vec{n}, \vec{n})\right] + N \int d\Omega_{\vec{n}'}\, A(\vec{n}', \vec{n}) \rho(\vec{r}, \vec{n}') A^+(\vec{n}', \vec{n}) + D(\vec{n}) \tag{33}$$

where $\rho$ is the density matrix (see equation (4)) and the matrix $A$ describes the transformation of the electric field $\vec{E} = E_1 \vec{l}_1 + E_2 \vec{l}_2$ due to light interaction with an elementary volume of a scattering medium in the coordinate system attached to a light-scattering layer. The matrix $D(\vec{n})$ can be obtained from the source term $B(\vec{n})$ in equation (32), assuming that $\vec{n} \| \vec{z}$. It follows after substitution of equation (5) into equation (33) and simple algebraic calculations that

$$\begin{aligned}
(\vec{n} \cdot \vec{\nabla}) I(\vec{r}, \vec{n}) &= -\varepsilon_{11} I(\vec{r}, \vec{n}) - \varepsilon_{12} Q(\vec{r}, \vec{n}) - \varepsilon_{13} U(\vec{r}, \vec{n}) - \varepsilon_{14} V(\vec{r}, \vec{n}) \\
&\quad + N \int d\Omega_{\vec{n}'} [\sigma_{11} I(\vec{r}, \vec{n}') + \sigma_{12} Q(\vec{r}, \vec{n}') + \sigma_{13} U(\vec{r}, \vec{n}') + \sigma_{14} V(\vec{r}, \vec{n}')] + \tilde{D}_1 \\
(\vec{n} \cdot \vec{\nabla}) Q(\vec{r}, \vec{n}) &= -\varepsilon_{21} I(\vec{r}, \vec{n}) - \varepsilon_{22} Q(\vec{r}, \vec{n}) - \varepsilon_{23} U(\vec{r}, \vec{n}) - \varepsilon_{24} V(\vec{r}, \vec{n}) \\
&\quad + N \int d\Omega_{\vec{n}'} [\sigma_{21} I(\vec{r}, \vec{n}') + \sigma_{22} Q(\vec{r}, \vec{n}') + \sigma_{23} U(\vec{r}, \vec{n}') + \sigma_{24} V(\vec{r}, \vec{n}')] + \tilde{D}_2 \\
(\vec{n} \cdot \vec{\nabla}) U(\vec{r}, \vec{n}) &= -\varepsilon_{31} I(\vec{r}, \vec{n}) - \varepsilon_{32} Q(\vec{r}, \vec{n}) - \varepsilon_{33} U(\vec{r}, \vec{n}) - \varepsilon_{34} V(\vec{r}, \vec{n}) \\
&\quad + N \int d\Omega_{\vec{n}'} [\sigma_{31} I(\vec{r}, \vec{n}') + \sigma_{32} Q(\vec{r}, \vec{n}') + \sigma_{33} U(\vec{r}, \vec{n}') + \sigma_{34} V(\vec{r}, \vec{n}')] + \tilde{D}_3 \\
(\vec{n} \cdot \vec{\nabla}) V(\vec{r}, \vec{n}) &= -\varepsilon_{41} I(\vec{r}, \vec{n}) - \varepsilon_{42} Q(\vec{r}, \vec{n}) - \varepsilon_{43} U(\vec{r}, \vec{n}) - \varepsilon_{44} V(\vec{r}, \vec{n}) \\
&\quad + N \int d\Omega_{\vec{n}'} [\sigma_{41} I(\vec{r}, \vec{n}') + \sigma_{42} Q(\vec{r}, \vec{n}') + \sigma_{43} U(\vec{r}, \vec{n}') + \sigma_{44} V(\vec{r}, \vec{n}')] + \tilde{D}_4
\end{aligned} \tag{34}$$

where expressions for elements $\sigma_{ij}$, $\varepsilon_{ij}$ are presented in the appendix and $\tilde{D}_1 = D_{11}+D_{22}$, $\tilde{D}_2 = D_{11}-D_{22}$, $\tilde{D}_3 = D_{12}+D_{21}$, $\tilde{D}_4 = \mathrm{i}(D_{12} - D_{21})$. Equations (34) can be rewritten in the vector form

$$(\vec{n}\cdot\vec{\nabla})\vec{S}(\vec{r},\vec{n}) = -\hat{\varepsilon}\vec{S}(\vec{r},\vec{n}) + N\int \mathrm{d}\Omega_{\vec{n}'}\,\hat{\sigma}\vec{S}(\vec{r},\vec{n}') + \vec{\tilde{D}}(\vec{r},\vec{n}) \tag{35}$$

where $\vec{\tilde{D}}(\vec{r},\vec{n})$ is the vector with elements $\tilde{D}_1$, $\tilde{D}_2$, $\tilde{D}_3$, $\tilde{D}_4$ and matrices $\hat{\varepsilon}$, $\hat{\sigma}$ have elements $\varepsilon_{ij}, \sigma_{ij}$ $(i, j = 1, 2, 3, 4)$, respectively. Equation (35) coincides with the standard vector radiative transfer equation [1]. It follows from this equation, within the framework of the scalar approximation, that

$$(\vec{n}\cdot\vec{\nabla})I(\vec{r},\vec{n}) = -\varepsilon I(\vec{r},\vec{n}) + N\int \mathrm{d}\Omega_{\vec{n}'}\,\sigma(\vec{n}',\vec{n})I\left(\vec{r},\vec{n}'\right) + B(\vec{n}) \tag{36}$$

where $\varepsilon = \varepsilon_{11}$, $\sigma = \sigma_{11}$. This is the well known scalar radiative transfer equation [2].

## 4. Conclusion

In conclusion, we underline that the derived equations (28) and (32) can be used to study the polarization characteristics of a light field in different complex random media, including gyrotropic and anisotropic ones, within the framework of the coordinate-free approach. They also could be of help in derivations of simple and compact tensor expressions for isotropic media.

The description of the interaction of light with matter (reflection, refraction and scattering of waves) in terms of the light beam tensor provides the possibility of considering the corresponding processes by covariant tensor methods, which in many cases simplifies the analytical derivations considerably. This was shown by Fedorov [4, 5] for the case of uniform anisotropic and gyrotropic media, where the selection of a coordinate system either related to the geometry of the problem (e.g. a plane-parallel layer) or to the symmetry of the dielectric tensor leads to extremely cumbersome results. It is expected that the same progress can be achieved for random complex media. Equations (32), (33) and (35) differ due either to the different coordinate systems or to the different description of the light beam. However, they provide the same information in the end, namely the intensity of the light, the degree of polarization and the characteristics of the polarization ellipse of reflected, transmitted and internal light fields. The choice of which equation to solve is largely dependent on the problem at hand.

For instance, radiative transfer in isotropic random media (e.g. fogs and clouds) can be studied with equation (35). Equation (32) simplifies radiative transport studies in complex media (e.g. asymmetric and anisotropic ones).

To solve equation (32) one needs to know the $3 \times 3$ scattering tensor $f(\vec{n},\vec{n}')$ of a random medium which depends on the size of the scatterers, their shape, the dielectric tensors of the particles, etc. This tensor can vary with the location inside a medium. Many particles can contribute to the value of $f(\vec{n}',\vec{n})$. Thus, the value of $f(\vec{n},\vec{n}')$ is the average value of the scattering tensor for the ensemble of scatterers.

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## Appendix. Extinction and scattering matrices

Elements of extinction $\hat{\varepsilon}$ and scattering $\hat{\sigma}$ matrices of the vector radiative transfer equation (35) are related to elements of matrix $A$ in equation (33) with the following formulae:

$$
\begin{aligned}
\varepsilon_{11} &= \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{44} = \lambda N(A''_{11}(0) + A''_{22}(0)) \\
\varepsilon_{12} &= \varepsilon_{21} = \lambda N(A''_{11}(0) - A''_{22}(0)) \\
\varepsilon_{13} &= \varepsilon_{31} = \lambda N(A''_{12}(0) + A''_{21}(0)) \\
\varepsilon_{14} &= \varepsilon_{41} = \lambda N(-A''_{12}(0) + A''_{21}(0)) \\
\varepsilon_{23} &= -\varepsilon_{32} = \lambda N(A''_{12}(0) - A''_{21}(0)) \\
\varepsilon_{24} &= -\varepsilon_{42} = -\lambda N(A'_{12}(0) + A'_{21}(0)) \\
\varepsilon_{34} &= -\varepsilon_{43} = \lambda N(A'_{22}(0) - A'_{11}(0)) \\
\sigma_{11} &= \tfrac{1}{2}\left(|A_{11}|^2 + |A_{22}|^2 + |A_{12}|^2 + |A_{21}|^2\right) \\
\sigma_{12} &= \tfrac{1}{2}\left(|A_{11}|^2 - |A_{12}|^2 + |A_{21}|^2 - |A_{22}|^2\right) \\
\sigma_{13} &= A'_{11}A'_{12} + A'_{21}A'_{22} + A''_{21}A''_{22} + A''_{11}A''_{12} \\
\sigma_{14} &= -A'_{21}A''_{22} + A''_{21}A'_{22} + A''_{11}A'_{12} - A'_{11}A''_{12} \\
\sigma_{21} &= \tfrac{1}{2}\left(-|A_{22}|^2 + |A_{11}|^2 + |A_{12}|^2 - |A_{21}|^2\right) \\
\sigma_{22} &= \tfrac{1}{2}\left(|A_{22}|^2 + |A_{11}|^2 - |A_{12}|^2 - |A_{21}|^2\right) \\
\sigma_{23} &= A''_{11}A''_{12} - A''_{21}A''_{22} - A'_{21}A'_{22} + A'_{11}A'_{12} \\
\sigma_{24} &= -A'_{11}A''_{12} + A''_{11}A'_{12} - A''_{21}A'_{22} + A'_{21}A''_{22} \\
\sigma_{31} &= A'_{21}A'_{11} + A''_{22}A''_{12} + A''_{21}A''_{11} + A'_{22}A'_{12} \\
\sigma_{32} &= -A''_{22}A''_{12} + A''_{21}A''_{11} + A'_{21}A'_{11} - A'_{22}A'_{12} \\
\sigma_{33} &= A''_{22}A''_{11} + A''_{21}A''_{12} + A'_{22}A'_{11} + A'_{21}A'_{12} \\
\sigma_{34} &= A'_{22}A''_{11} - A''_{22}A'_{11} - A'_{21}A''_{12} + A''_{21}A'_{12} \\
\sigma_{41} &= -A'_{22}A''_{12} + A''_{22}A'_{12} - A'_{21}A''_{11} + A''_{21}A'_{11} \\
\sigma_{42} &= A'_{22}A''_{12} - A''_{22}A'_{12} - A'_{21}A''_{11} + A''_{21}A'_{11} \\
\sigma_{43} &= -A'_{22}A''_{11} + A''_{22}A'_{11} - A'_{21}A''_{12} + A''_{21}A'_{12} \\
\sigma_{44} &= \left[A'_{22}A'_{11} + A''_{22}A''_{11} - A'_{21}A'_{12} - A''_{21}A''_{12}\right]
\end{aligned}
$$

where $A'_{ij} = \mathrm{Re}(A_{ij})$, $A''_{ij} = \mathrm{Im}(A_{ij})$. The symbol $A_{ij}(0)$ denotes the value of the amplitude matrix element in the forward scattering direction.

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